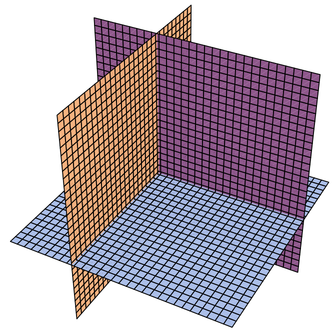
附录 A　常用广义正交曲线坐标系

笛卡尔直角坐标系

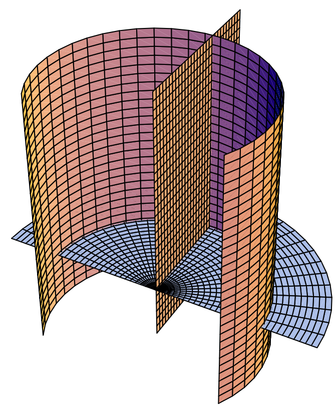
$$(x, y, z) \qquad \begin{cases} h_x = 1 \\ h_y = 1 \\ h_z = 1 \end{cases}$$



圆柱坐标系

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases} \quad \begin{cases} h_\rho = 1 \\ h_\theta = \rho \\ h_z = 1 \end{cases}$$

$$\begin{pmatrix} \hat{\boldsymbol{\rho}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{z}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

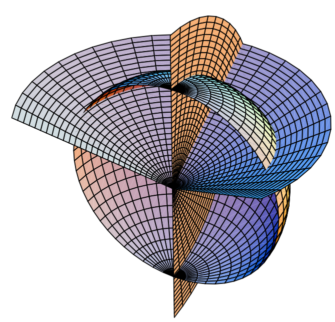


球坐标系

$$\begin{cases} x = \rho\cos\varphi\sin\theta \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\theta \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \theta = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \\ \varphi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

$$\begin{cases} h_\rho = 1 \\ h_\theta = \rho \\ h_\varphi = \rho\sin\theta \end{cases}$$

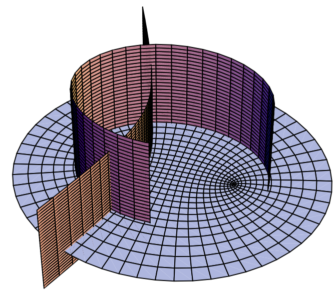
$$\begin{pmatrix} \hat{\boldsymbol{\rho}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\cos\varphi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$



椭圆柱坐标系

$$\begin{cases} x = a\cos v\cosh u \\ y = a\sin v\sinh u \\ z = z \end{cases} \qquad \begin{cases} u = \Re\left(\cosh^{-1}\left(\dfrac{x+\mathrm{i}y}{a}\right)\right) \\ v = \Im\left(\cosh^{-1}\left(\dfrac{x+\mathrm{i}y}{a}\right)\right) \\ z = z \end{cases}$$

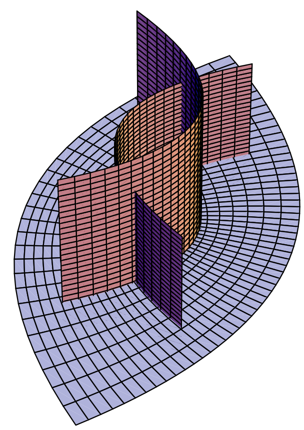
$$\begin{cases} h_u = a\sqrt{\sin^2 v + \sinh^2 u} \\ h_v = a\sqrt{\sin^2 v + \sinh^2 u} \\ h_z = 1 \end{cases}$$



$$\begin{pmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{z}} \end{pmatrix} = \begin{pmatrix} a\cos v\sinh u & -a\cosh u\sin v & 0 \\ a\cosh u\sin v & a\cos v\sinh u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

抛物柱坐标系

$$\begin{cases} x = \dfrac{1}{2}\left(u^2 - v^2\right) \\ y = uv \\ z = z \end{cases} \qquad \begin{cases} u = \dfrac{y}{\sqrt{\sqrt{x^2+y^2}-x}} \\ v = \sqrt{\sqrt{x^2+y^2}-x} \\ z = z \end{cases} \qquad \begin{cases} h_u = \sqrt{u^2+v^2} \\ h_v = \sqrt{u^2+v^2} \\ h_z = 1 \end{cases}$$

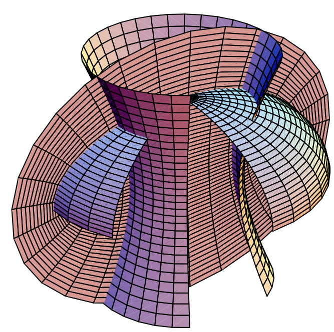


$$\begin{pmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{z}} \end{pmatrix} = \begin{pmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

扁椭球坐标系

$$\begin{cases} x = a\cos\xi\cos\varphi\cosh\eta \\ y = a\cos\xi\cosh\eta\sin\varphi \\ z = a\sin\xi\sinh\eta \end{cases} \qquad \begin{cases} \xi = \Im\left(\cosh^{-1}\left(\dfrac{\mathrm{i}z+\sqrt{x^2+y^2}}{a}\right)\right) \\ \eta = \Re\left(\cosh^{-1}\left(\dfrac{\mathrm{i}z+\sqrt{x^2+y^2}}{a}\right)\right) \\ \varphi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

$$\begin{cases} h_\xi = a\sqrt{\sin^2\xi + \sinh^2\eta} \\ h_\eta = a\sqrt{\sin^2\xi + \sinh^2\eta} \\ h_\varphi = a\cos\xi\cosh\eta \end{cases}$$

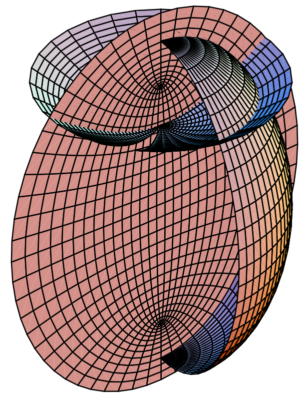


$$\begin{pmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\eta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} -a\cos\varphi\cosh\eta\sin\xi & a\cos\xi\cos\varphi\sinh\eta & -a\cos\xi\cosh\eta\sin\varphi \\ -a\cosh\eta\sin\xi\sin\varphi & a\cos\xi\sin\varphi\sinh\eta & a\cos\xi\cos\varphi\cosh\eta \\ a\cos\xi\sinh\eta & a\cosh\eta\sin\xi & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

长椭球坐标系

$$\begin{cases} x = a\cos\varphi\sin\eta\sinh\xi \\ y = a\sin\eta\sin\varphi\sinh\xi \\ z = a\cos\eta\cosh\xi \end{cases}$$

$$\begin{cases} \xi = \Re\left(\cosh^{-1}\left(\frac{z+\mathrm{i}\sqrt{x^2+y^2}}{a}\right)\right) \\ \eta = \Im\left(\cosh^{-1}\left(\frac{z+\mathrm{i}\sqrt{x^2+y^2}}{a}\right)\right) \\ \varphi = \tan^{-1}\frac{y}{x} \end{cases}$$



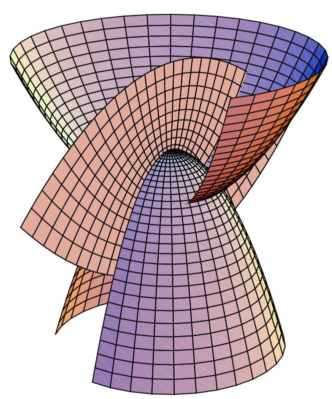
$$\begin{cases} h_\xi = a\sqrt{\sin^2\eta+\sinh^2\xi} \\ h_\eta = a\sqrt{\sin^2\eta+\sinh^2\xi} \\ h_\varphi = a\sin\eta\sinh\xi \end{cases}$$

$$\begin{pmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\eta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} a\cos\varphi\cosh\xi\sin\eta & a\cos\eta\cos\varphi\sinh\xi & -a\sin\eta\sin\varphi\sinh\xi \\ a\cosh\xi\sin\eta\sin\varphi & a\cos\eta\sin\varphi\sinh\xi & a\cos\varphi\sin\eta\sinh\xi \\ a\cos\eta\sinh\xi & -a\cosh\xi\sin\eta & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

旋转抛物面坐标系

$$\begin{cases} x = uv\cos\varphi \\ y = uv\sin\varphi \\ z = \frac{1}{2}\left(u^2-v^2\right) \end{cases}$$

$$\begin{cases} u = \frac{\sqrt{x^2+y^2}}{\sqrt{\sqrt{x^2+y^2+z^2}-z}} \\ v = \sqrt{\sqrt{x^2+y^2+z^2}-z} \\ \varphi = \tan^{-1}\frac{y}{x} \end{cases}$$



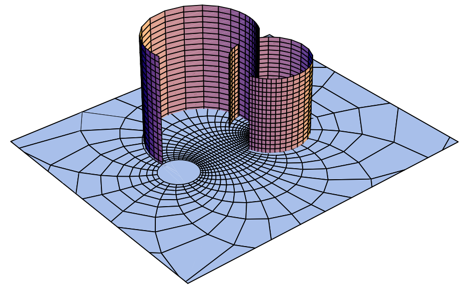
$$\begin{cases} h_u = \sqrt{u^2+v^2} \\ h_v = \sqrt{u^2+v^2} \\ h_\varphi = uv \end{cases}$$

$$\begin{pmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{z}} \end{pmatrix} = \begin{pmatrix} v\cos\varphi & u\cos\varphi & -uv\sin\varphi \\ v\sin\varphi & u\sin\varphi & uv\cos\varphi \\ u & -v & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

双极坐标系

$$\begin{cases} x = \dfrac{a\sinh v}{\cosh v - \cos u} \\ y = \dfrac{a\sin u}{\cosh v - \cos u} \\ z = z \end{cases} \qquad \begin{cases} u = -2\Im\left(\coth^{-1}\left(\dfrac{x+\mathrm{i}y}{a}\right)\right) \\ v = 2\Re\left(\coth^{-1}\left(\dfrac{x+\mathrm{i}y}{a}\right)\right) \\ z = z \end{cases}$$

$$\begin{cases} h_u = \dfrac{a}{\cosh v - \cos u} \\ h_v = \dfrac{a}{\cosh v - \cos u} \\ h_z = 1 \end{cases}$$

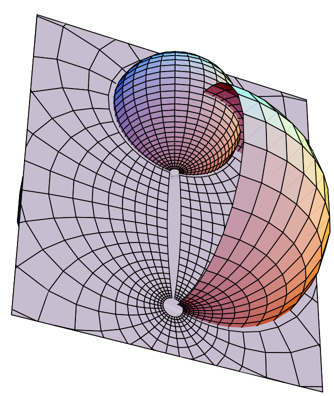


$$\begin{pmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{z}} \end{pmatrix} = \begin{pmatrix} -\dfrac{a\sin u\sinh v}{(\cosh v-\cos u)^2} & \dfrac{a\cosh v}{\cosh v-\cos u} - \dfrac{a\sinh^2 v}{(\cosh v-\cos u)^2} & 0 \\ \dfrac{a\cos u}{\cosh v-\cos u} - \dfrac{a\sin^2 u}{(\cosh v-\cos u)^2} & -\dfrac{a\sin u\sinh v}{(\cosh v-\cos u)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

双球坐标系

$$\begin{cases} x = \dfrac{a\cos\varphi\sin u}{\cosh v - \cos u} \\ y = \dfrac{a\sin u\sin\varphi}{\cosh v - \cos u} \\ z = \dfrac{a\sinh v}{\cosh v - \cos u} \end{cases} \qquad \begin{cases} h_u = \dfrac{a}{\cosh v - \cos u} \\ h_v = \dfrac{a}{\cosh v - \cos u} \\ h_\varphi = \dfrac{a\sin u}{\cosh v - \cos u} \end{cases}$$

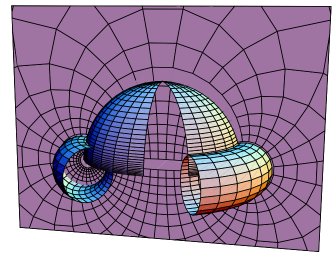
$$\begin{cases} u = -2\Im\left(\coth^{-1}\left(\dfrac{z+\mathrm{i}\sqrt{x^2+y^2}}{a}\right)\right) \\ v = 2\Re\left(\coth^{-1}\left(\dfrac{z+\mathrm{i}\sqrt{x^2+y^2}}{a}\right)\right) \\ \varphi = \tan^{-1}\dfrac{y}{x} \end{cases}$$



$$\begin{pmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \dfrac{a\cos u\cos\varphi}{\cosh v-\cos u} - \dfrac{a\cos\varphi\sin^2 u}{(\cosh v-\cos u)^2} & -\dfrac{a\cos\varphi\sin u\sinh v}{(\cosh v-\cos u)^2} & -\dfrac{a\sin u\sin\varphi}{\cosh v-\cos u} \\ \dfrac{a\cos u\sin\varphi}{\cosh v-\cos u} - \dfrac{a\sin^2 u\sin\varphi}{(\cosh v-\cos u)^2} & -\dfrac{a\sin u\sin\varphi\sinh v}{(\cosh v-\cos u)^2} & \dfrac{a\cos\varphi\sin u}{\cosh v-\cos u} \\ -\dfrac{a\sin u\sinh v}{(\cosh v-\cos u)^2} & \dfrac{a\cosh v}{\cosh v-\cos u} - \dfrac{a\sinh^2 v}{(\cosh v-\cos u)^2} & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

环坐标系

$$\begin{cases} x = \dfrac{a\cos\varphi\sinh u}{\cosh u - \cos v} \\ y = \dfrac{a\sin\varphi\sinh u}{\cosh u - \cos v} \\ z = \dfrac{a\sin v}{\cosh u - \cos v} \end{cases} \qquad \begin{cases} u = 2\Re\left(\coth^{-1}\left(\dfrac{\mathrm{i}z + \sqrt{x^2+y^2}}{a}\right)\right) \\ v = -2\Im\left(\coth^{-1}\left(\dfrac{\mathrm{i}z + \sqrt{x^2+y^2}}{a}\right)\right) \\ \varphi = \tan^{-1}\dfrac{y}{x} \end{cases}$$



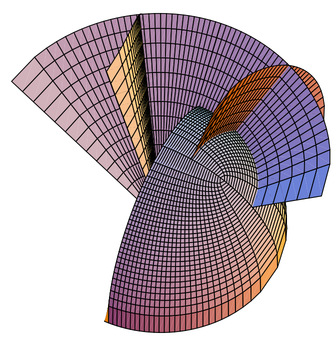
$$\begin{cases} h_u = \dfrac{a}{\cosh u - \cos v} \\ h_v = \dfrac{a}{\cosh u - \cos v} \\ h_\varphi = \dfrac{a\sinh u}{\cosh u - \cos v} \end{cases}$$

$$\begin{pmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \dfrac{a\cos\varphi\cosh u}{\cosh u - \cos v} - \dfrac{a\cos\varphi\sinh^2 u}{(\cosh u - \cos v)^2} & -\dfrac{a\cos\varphi\sin v\sinh u}{(\cosh u - \cos v)^2} & -\dfrac{a\sin\varphi\sinh u}{\cosh u - \cos v} \\ \dfrac{a\cosh u\sin\varphi}{\cosh u - \cos v} - \dfrac{a\sin\varphi\sinh^2 u}{(\cosh u - \cos v)^2} & -\dfrac{a\sin v\sin\varphi\sinh u}{(\cosh u - \cos v)^2} & \dfrac{a\cos\varphi\sinh u}{\cosh u - \cos v} \\ -\dfrac{a\sin v\sinh u}{(\cosh u - \cos v)^2} & \dfrac{a\cos v}{\cosh u - \cos v} - \dfrac{a\sin^2 v}{(\cosh u - \cos v)^2} & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

圆锥坐标系

参数 $\{0 < a < b < \infty\}$ $\begin{pmatrix} \lambda & \mu & \nu \\ -\infty < \lambda < \infty & a^2 < \mu^2 < b^2 & \nu^2 < a^2 \end{pmatrix}$

$$\begin{cases} x = \dfrac{\lambda|\mu\nu|}{ab} \\ y = \dfrac{1}{a}\sqrt{\dfrac{(\mu^2-a^2)(\nu^2-a^2)}{a^2-b^2}}|\lambda|\operatorname{sgn}(\mu) \\ z = \dfrac{1}{b}\sqrt{-\dfrac{(\mu^2-b^2)(\nu^2-b^2)}{a^2-b^2}}|\lambda|\operatorname{sgn}(\nu) \end{cases}$$

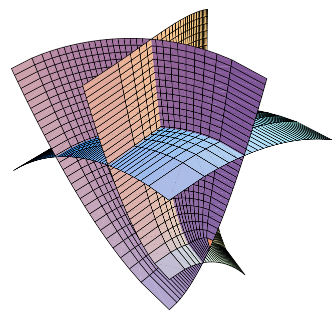


$$\begin{cases} h_\lambda = 1 \\ h_\mu = \dfrac{\sqrt{\mu^2-\nu^2}|\lambda|}{\sqrt{b^2-\mu^2}\sqrt{\mu^2-a^2}} \\ h_\nu = \dfrac{\sqrt{\mu^2-\nu^2}|\lambda|}{\sqrt{\nu^2-a^2}\sqrt{\nu^2-b^2}} \end{cases}$$

$$\begin{cases} \lambda = \sqrt{x^2+y^2+z^2}\operatorname{sgn}(x) \\ \mu = \dfrac{\operatorname{sgn}(y)}{\sqrt{2}}\sqrt{\dfrac{a^2x^2+b^2x^2+b^2y^2+a^2z^2+\sqrt{(-a^2x^2-b^2x^2-b^2y^2-a^2z^2)^2-4a^2b^2x^2(x^2+y^2+z^2)}}{x^2+y^2+z^2}} \\ \nu = \dfrac{\operatorname{sgn}(z)}{\sqrt{2}}\sqrt{\dfrac{a^2x^2+b^2x^2+b^2y^2+a^2z^2-\sqrt{(-a^2x^2-b^2x^2-b^2y^2-a^2z^2)^2-4a^2b^2x^2(x^2+y^2+z^2)}}{x^2+y^2+z^2}} \end{cases}$$

共焦抛物面坐标系

$$\text{参数}\ \{0<b<a<\infty\}\begin{pmatrix}\lambda & \mu & \nu\\ -\infty<\lambda<b^2 & b^2<\mu<a^2 & a^2<\nu<\infty\end{pmatrix}$$

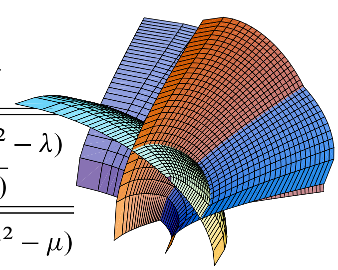


$$\begin{cases}x=\sqrt{\dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{b^2-a^2}}\\ y=\sqrt{\dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{a^2-b^2}}\\ z=\dfrac{1}{2}(a^2+b^2-\lambda-\mu-\nu)\end{cases}\qquad \begin{cases}h_\lambda=\dfrac{\sqrt{(\mu-\lambda)(\nu-\lambda)}}{2\sqrt{(a^2-\lambda)(b^2-\lambda)}}\\ h_\mu=\dfrac{\sqrt{(\lambda-\mu)(\nu-\mu)}}{2\sqrt{(a^2-\mu)(b^2-\mu)}}\\ h_\nu=\dfrac{\sqrt{(\lambda-\nu)(\mu-\nu)}}{2\sqrt{(a^2-\nu)(b^2-\nu)}}\end{cases}$$

$$\begin{pmatrix}\hat{\boldsymbol\lambda}\\ \hat{\boldsymbol\mu}\\ \hat{\boldsymbol\nu}\end{pmatrix}=\begin{pmatrix}-\dfrac{\sqrt{\frac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{b^2-a^2}}}{2\,(a^2-\lambda)} & -\dfrac{\sqrt{\frac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{b^2-a^2}}}{2\,(a^2-\mu)} & -\dfrac{\sqrt{\frac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{b^2-a^2}}}{2\,(a^2-\nu)}\\ -\dfrac{\sqrt{\frac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{a^2-b^2}}}{2\,(b^2-\lambda)} & -\dfrac{\sqrt{\frac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{a^2-b^2}}}{2\,(b^2-\mu)} & -\dfrac{\sqrt{\frac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{a^2-b^2}}}{2\,(b^2-\nu)}\\ -\dfrac{1}{2} & -\dfrac{1}{2} & -\dfrac{1}{2}\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol x}\\ \hat{\boldsymbol y}\\ \hat{\boldsymbol z}\end{pmatrix}$$

共焦椭球面坐标系

$$\text{参数}\ \{0<c<b<a<\infty\}\begin{pmatrix}\lambda & \mu & \nu\\ -\infty<\lambda<c^2 & c^2<\mu<b^2 & b^2<\nu<a^2\end{pmatrix}$$



$$\begin{cases}x=\sqrt{\dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{b^2-a^2}}\\ y=\sqrt{\dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{a^2-b^2}}\\ z=\dfrac{1}{2}(a^2+b^2-\lambda-\mu-\nu)\end{cases}\qquad \begin{cases}h_\lambda=\dfrac{\sqrt{(\mu-\lambda)(\nu-\lambda)}}{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}\\ h_\mu=\dfrac{\sqrt{(\lambda-\mu)(\nu-\mu)}}{2\sqrt{(a^2-\mu)(b^2-\mu)(c^2-\mu)}}\\ h_\nu=\dfrac{\sqrt{(\lambda-\nu)(\mu-\nu)}}{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}\end{cases}$$

$$\begin{pmatrix}\hat{\lambda}\\ \hat{\mu}\\ \hat{\nu}\end{pmatrix}=\begin{pmatrix}-\frac{\sqrt{\frac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)}}}{2(a^2-\lambda)} & -\frac{\sqrt{\frac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)}}}{2(a^2-\mu)} & -\frac{\sqrt{\frac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)}}}{2(a^2-\nu)}\\ -\frac{\sqrt{\frac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{(b^2-a^2)(b^2-c^2)}}}{2(b^2-\lambda)} & -\frac{\sqrt{\frac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{(b^2-a^2)(b^2-c^2)}}}{2(b^2-\mu)} & -\frac{\sqrt{\frac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{(b^2-a^2)(b^2-c^2)}}}{2(b^2-\nu)}\\ -\frac{\sqrt{\frac{(c^2-\lambda)(c^2-\mu)(c^2-\nu)}{(a^2-c^2)(b^2-c^2)}}}{2(c^2-\lambda)} & -\frac{\sqrt{\frac{(c^2-\lambda)(c^2-\mu)(c^2-\nu)}{(a^2-c^2)(b^2-c^2)}}}{2(c^2-\mu)} & -\frac{\sqrt{\frac{(c^2-\lambda)(c^2-\mu)(c^2-\nu)}{(a^2-c^2)(b^2-c^2)}}}{2(c^2-\nu)}\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix}$$